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Integral Global Minimization: Algorithms, Implementations and Numerical Tests

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Abstract. The theoretical foundation of integral global optimization has become widely known and well accepted [4],[24],[25]. However, more effort is needed to demonstrate the effectiveness of the integral global optimization algorithms. In this work we detail the implementation of the integral global minimization algorithms. We describe how the integral global optimization method handles nonconvex unconstrained or box constrained, constrained or discrete minimization problems. We illustrate the flexibility and the efficiency of integral global optimization method by presenting the performance of algorithms on a collection of well known test problems in global optimization literature. We provide the software which solves these test problems and other minimization problems. The performance of the computations demonstrates that the integral global algorithms are not only extremely flexible and reliable but also very efficient.

Keywords: Integral global minimization, Monte Carlo implementation, test problems, discontinuous penalty method, robustification

1. Introduction

Let X be a topological space, $f : X \to R^1$ a function and S a subset of X. The problem considered here is to find the infimum of f over S

$$c^* = \inf_{x \in S} f(x) \tag{1}$$

and the set of global minimizers

$$H^* = \{x \in S : f(x) = c^*\}, \tag{2}$$

if H^* is nonempty.

Most of the conventional optimization theory and methods are gradient-based. They can only be applied to characterize and to find a local minimizer of an objective function. The gradient based iterative algorithms, which are easy to implement, usually have higher convergence rates. The gradient-based theory and methods are the main stream of the research in optimization. However, in many applications, it

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is often more desirable to find a global minimizer than to find a local one, especially when we deal with a nonconvex optimization problem.

An integral approach of global optimization has been developed to deal with nonconvex minimization problems of a class of discontinuous objective functions (see [4], [30], [31]). Integral global optimization algorithms are implemented by properly designed Monte-Carlo techniques. In this work we describe the techniques of the implementations of the algorithms. We also present the performance of the algorithms on a collection of well known test problems. A companion diskette containing all the software necessary for solving unconstrained or constrained minimization problems presented in this paper on an MS-DOS environment is available upon the request to the authors.

The following is the organization of the paper. In Section 2, we describe briefly the main ideas of the integral global optimization theory. Section 3 is devoted to the detailed explanation of the implementation of integral global minimization algorithms for simple unconstrained models. Some statistical analysis of the implementation is also presented in Section 3. More implementation techniques are discussed in Section 4. In Section 5, we consider constrained and discrete or mixed problems. A collection of test problems from global optimization literature are solved by the integral global minimization algorithm in Section 6.

2. Integral Global Optimization

We summarize the main ideas of the integral global minimization theory. The reader is referred to [4], [30], [31] for details.

Optimality Conditions.

Recall that a set D in a topological space X is *robust* iff

$$\mathrm{cl}\ D = \mathrm{cl}\ \mathrm{int}\ D. \tag{3}$$

A function $f : X \to R^1$ is *upper robust* over S iff the set

$$F_c = \{x \in S : f(x) < c\} \tag{4}$$

is robust for each real number c. Upper robustness of a function generalizes the concepte of continuity of a function. Based on such a generalization, a unified approach to continuous, discrete and mixed minimization problems, integral global optimization, is established.

For the problem (1) under the assumptions that f is lower semicontinuous and upper robust; (X, Ω, μ) is a Q-measure space (the measure μ have a property that the measure of a nonempty open set is positive); $S \subset X$ is robust and there is a real number b such that $\{x \in S : f(x) \leq b\}$ is compact, the following statements are equivalent:

1. A point $x^* \in S$ is a global minimizer and $c^* = f(x^*)$ is the corresponding global minimum value;

2. $M(f, c^*; S) = c^*$ (mean value condition);

3. $V_1(f, c^*; S) = 0$ (modified variance condition),

where

$$M(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} f(x) d\mu \tag{5}$$

and

$$V_1(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c} (f(x) - c)^2 d\mu \tag{6}$$

are the *mean value* and *modified variance*, respectively, of f over its level set

$$H_c = \{x : f(x) \leq c\}. \tag{7}$$

The Algorithm.

Step 1 : Take $c_0 > c^*$ and $\epsilon > 0$; $k := 0$;

Step 2 : $c_{k+1} := M(f, c_k; S)$; $v_{k+1} := V_1(f, c_k; S)$; $H_{k+1} \cap S := \{x \in S : f(x) \leq c_{k+1}\}$;

Step 3 : If $v_{k+1} \geq \epsilon$ then $k := k + 1$; go to Step 2;

Step 4 : $c^* \Longleftarrow c_{k+1}$; $H^* \Longleftarrow H_{c_{k+1}} \cap S$; Stop.

If we take $\epsilon = 0$, then we obtain two monotone sequences:

$$c_0 \geq c_1 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \tag{8}$$

and

$$H_{c_0} \cap S \supset H_{c_1} \cap S \supset \cdots \supset H_{c_k} \cap S \supset H_{c_{k+1}} \cap S \supset \cdots. \tag{9}$$

Let

$$c^* = \lim_{k \to \infty} c_k \quad \text{and} \quad H^* = \bigcap_{k=1}^{\infty} H_{c_k} \cap S, \tag{10}$$

then c^* is the global minimum value of f over S and H^* is the set of global minimizers.

From the above algorithm, we realize that the integral method for finding global minimizers requires the computation of a sequence of mean values and modified variances, and a sequence of level sets. Finding a mean value and modified variance are equivalent to computing integrals of a function of several variables; the determination of a level set is, in general, more involved. This suggests that a

Monte-Carlo based technique for finding global minimizers is appropriate. The error of integration by the Monte Carlo method is proportional to $\sigma/\sqrt{t}$, where t is the number of samples and σ^2 is the variance of sample distribution. Note that the accuracy at early steps of the algorithm is not generally required. since σ^2 will tend to zero as the mean value goes to the global minimum value (the modified variance condition), the Monte Carlo approximation will become more accurate near the global minimum value even though the number t of random samples is not very large.

In next section, we will discuss the Monte Carlo implementation of the algorithme.

3. Monte-Carlo Implementation of a Simple Model

Let us first consider a simple model of a global minimization problem. Suppose that the constraint set D is a cuboid in R^n,

$$D = \{x : a^i \leq x^i \leq b^i,\ i = 1, \ldots, n\} \tag{11}$$

and the objective function f is a lower semicontinuous and upper robust function with a *unique* global minimizer $x^* \in D$. In other words, for a decreasing sequence $\{c_k\}$ which converges to the global minimum value c^*, the size of the level sets satisfies:

$$\rho_k = \rho(H_{c_k}) = \max_{x,y \in H_{c_k}} \|x - y\| \to 0 \text{ as } k \to \infty. \tag{12}$$

We then have

$$c^* = \min_{x \in D} f(x) = \min_{x \in H_{c_k} \cap D} f(x) = \min_{x \in D_k} f(x), \tag{13}$$

where D_k is the smallest cuboid containing the level set $H_{c_k} \cap D$.

Instead of computing $M(f, c_k; D)$ and $V_1(f, c_k; D)$ in the algorithm in the previous section, we compute $M(f, c_k; D_k)$ and $V_1(f, c_k; D_k)$ at each iteration. The following is an algorithm for this model:

Step 1 : Take $c_0 > \min_{x \in D} f(x)$. Let $D_0 = D$ be an initial cuboid. Set $k = 0$.

Step 2 : Compute the mean value

$$c_{k+1} = M(f, c_k; D_k) = \frac{1}{\mu(H_{c_k} \cap D_k)} \int_{H_{c_k} \cap D_k} f(x) d\mu,$$

where D_k be the smallest closed cuboid containing the level set $H_{c_k} = \{\, x : f(x) \leq c_k \,\}$.

Step 3 : Compute the modified variance

$$v_f = V_1(f, c_k; D_k) = \frac{1}{\mu(H_{c_k} \cap D_k)} \int_{H_{c_k} \cap D_k} (f(x) - c_k)^2 d\mu.$$

Step 4 : If $v_f \geq \epsilon$, set $k := k + 1$, and go to Step 2; otherwise, go to Step 5.

Step 5 : Let $c^* \Leftarrow c_{k+1}$ and $H^* \Leftarrow H_{c_{k+1}}$. Stop.

At each iteration, we try to find D_k instead of level set H_{c_k}, where

$$\begin{aligned} D_k &= \{x : a_k^i \leq x^i \leq b_k^i,\ i = 1, \dots, n\}, \\ a_k^i &= \min\{x_i : (x^1, \dots, x^i, \dots, x^n) \in H_{c_k}\}, \\ b_k^i &= \max\{x_i : (x^1, \dots, x^i, \dots, x^n) \in H_{c_k}\}. \end{aligned}$$

Let $\epsilon = 0$. The above algorithm produces a sequence of level constants $\{c_k\}$ and a sequence of cuboid $\{D_k\}$.

LEMMA 1 *For the foregoing simple model,*

$$\{x^*\} = \bigcap_{k=1}^{\infty} D_k, \tag{14}$$

where x^* is the unique global minimizer of the minimization problem.

Proof. By the definitions of the level set H_{c_k} and D_k, $x^* \in H_{c_k} \cap D_k$, for each k. We have

$$x^* \in \bigcap_{k=1}^{\infty} (H_{c_k} \cap D_k) \subset \bigcap_{k=1}^{\infty} D_k.$$

It follows from (12) and the construction of D_k, the diameter of D_k approaches to 0. The Cantor theorem [2] applies. □

3.1. Monte Carlo Implementation

The implementation of the simple model can be described as follows:

1. Approximation of H_{c_0} and $M(f, c_o; D)$:

Let $\xi = (\xi^1, \dots, \xi^n)$ be an independent n-multiple random number which is uniformly distributed on $[0, 1]^n$. Let

$$x^i = a^i + (b^i - a^i) \cdot \xi^i,\ i = 1, \dots, n. \tag{15}$$

Then $x = (x^1, \dots, x^n)$ is uniformly distributed on D.

Take km samples and evaluate function values $f(x_j)$, $j = 1, 2, \dots, km$, at these sample points. Comparing the values of the function f at these points, we obtain a set W of sample points corresponding to the t smallest function values: $FV[j]$, $j = 1, 2, \ldots, t$, ordered by their values, i.e.,

$$FV[1] \geq FV[2] \geq \cdots \geq FV[t]. \tag{16}$$

The set W is called an *acceptance set* which can be regarded as an approximation to the level set H_{c_0}, where $c_0 = FV[1]$ is the largest value of $\{FV[j]\}$. The positive integer t is called the *statistical index*. It is clear that $f(x) \leq c_0$ for all $x \in W$. Also, the mean value of f over the level set H_{c_0} can be approximated by the mean value of $\{FV[j]\}$:

$$c_1 = M(f, c_0; D) \approx (FV[1] + \cdots + FV[t])/t. \tag{17}$$

2. Generating a new cuboid by W:

The new cuboid domain of dimension n

$$D_1 = \{x = (x^1, \ldots, x^n) : a_1^i \leq x^i \leq b_1^i,\ i = 1, \ldots, n\} \tag{18}$$

can be generated by the following procedure. Suppose that the random samples in W are $\tau_1, \ldots, \tau_n$. Let

$$\sigma_0^i = \min(\tau_1^i, \ldots, \tau_n^i) \text{ and } \sigma_1^i = \max(\tau_1^i, \ldots, \tau_n^i),\ i = 1, \ldots, n, \tag{19}$$

where $\tau_j = (\tau_j^1, \ldots, \tau_j^n)$, $j = 1, \ldots, t$. We use

$$a^i = \sigma_0^i - \frac{\sigma_1^i - \sigma_0^i}{t-1} \text{ and } b^i = \sigma_1^i + \frac{\sigma_1^i - \sigma_0^i}{t-1} \tag{20}$$

as estimators to generate a_1^i and b_1^i, $i = 1, \ldots, n$.

3. Continuing the iterative process:

The samples are now taken in the new domain D_1. Take a random sample point $x = (x^1, \ldots, x^n)$ in D_1, where

$$x^i = a_1^i + (b_1^i - a_1^i) \cdot \xi^i,\ i = 1, \ldots, n. \tag{21}$$

Evaluate $f(x)$. If $f(x) \geq FV[1]$, then drop this sample point; otherwise, update the sets $\{FV[j]\}$ and W such that the new $\{FV[j]\}$ is made up of the t best function values obtained so far. The acceptance set W is updated accordingly. Repeating this procedure until $FV[1] \leq c_1$, we obtain, new FV and W.

4. Iterative solution:

At each iteration, the smallest value $FV[t]$ in the set $\{FV[j]\}$ and the corresponding point in W can be regarded as an iterative solution.

5. Convergence criterion:

The modified variance v_f of $\{FV[j]\}$, which is given by

$$v_f = \frac{1}{t-1} \sum_{j=2}^{t} (FV[j] - FV[1])^2, \tag{22}$$

can be regarded as an approximation of $V_1(f, c_k; D_k)$ at each iteration. If v_f is less than the given precision ϵ, then the iterative process terminates, and the current iteration in Step 4 would serve as an estimate of the global minimum value and the global minimizer.

4. More Techniques on Implementation

4.1. Adaptive Change of Search Sets

Consider a minimization problem

$$\min_{x \in S} f(x).$$

The adaptive change of search sets technique allows an initial choice of a computationally manageable set S_0 and then during the iteration process moves on to better performing sets S_k while still holding down their "size." The idea of this technique is to make a more perceptive use of the information generated from previous iterations to reduce the size of search sets.

Let c_0 be a real number and S_0 be an initial compact robust search set where $\mu(H_{c_0} \cap S) > 0$. Let

$$c_1 = M(f, c_0; S_0) = \frac{1}{\mu(H_{c_0} \cap S)} \int_{H_{c_0} \cap S} f(x) d\mu.$$

Then $c_0 \geq c_1 \geq c^* = \min_{x \in S} f(x)$. Take a robust set $S_1 \subset S$ such that $S_0 \cap H_{c_1} \subset S_1$, which implies that $S_0 \cap H_{c_1} \subset S_1 \cap H_{c_1}$.

Furthermore, we have

$$\mu(S_1 \cap H_{c_1}) \geq \mu(S_0 \cap H_{c_1}) > 0, \tag{23}$$

where $\mu(S_0 \cap H_{c_1}) > 0$ because $\mu(S_0 \cap H_{c_0}) > 0$. Let $c_2 = M(f, c_1; S_1)$.

In general, we require a set S_{k+1} be such that

$$S_{k-1} \cap H_{c_k} \subset S_k, \quad k = 1, 2, \ldots, \tag{24}$$

and let $c_{k+1} = M(f, c_k; S_k)$, $k = 0, 1, 2, \ldots$. In this manner we have constructed a sequence of robust search sets and obtain the following two sequences :

$$c_0 \geq c_1 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \tag{25}$$

and

$$H_{c_0} \supset H_{c_1} \supset \cdots \supset H_{c_k} \supset H_{c_{k+1}} \supset \cdots. \tag{26}$$

Denote

$$S_L = \bigcup_{k=1}^{\infty} S_k \quad \text{and} \quad G_L = \text{cl } S_L. \tag{27}$$

Sometimes the structures of sets S_k, $k = 0, 1, 2, \ldots$, are complicated, and a further assumption is required:

$$(SM): \qquad \mu(S_L) = \mu(\text{cl } S_L).$$

Let $c^* = \lim_{k \to \infty} c_k$ and $H^* = \lim_{k \to \infty} H_{c_k} = \bigcap_{k=1}^{\infty} H_{c_k}$.

THEOREM 1 *Under the assumptions (A), (M), and (SM), the limit c^* is the global minimum value and $H^* \cap G_L$ is the set of corresponding global minimizers of f over G_L.*

Optimality conditions of our change-of-set model can also be given. Since the search sets are changed step by step, the optimality conditions are described in limit forms. Suppose that $\{c_k\}$ is a decreasing sequence which tends to c^*, and $\{S_k\}$ is a sequence of robust sets such that

$$S_k \subset S \quad \text{and} \quad S_k \cap H_{c_{k+1}} \subset S_{k+1}, \; k = 0, 1, 2, \ldots. \tag{28}$$

THEOREM 2 *The following statements are equivalent:*

(*i*) c^* *is the global minimum value of f over G_L;*

(*ii*) $\displaystyle \lim_{k \to \infty} \frac{1}{\mu(S_k \cap H_{c_k})} \int_{S_k \cap H_{c_k}} f(x) d\mu = c^*;$

(*iii*) $\displaystyle \lim_{k \to \infty} \frac{1}{\mu(S_k \cap H_{c_k})} \int_{S_k \cap H_{c_k}} (f(x) - c^*)^2 d\mu = 0.$

A technique of reduction of the skew rate

$$\delta = \frac{2x^* - (a + b)}{b - a} \tag{29}$$

was proposed to reduce the amount of computation. Thus, we can adopt the following change-of-set strategy: *to move the search set in such directions so as to reduce the skew rate.*

Take three constant $\delta_0 \geq 0$, $\delta_1 > \delta_2 \geq 0$. The skew rate δ is considered not too large if $|\delta| \leq \delta_0$. In this case, the search domain need not be changed. If $\delta > \delta_0$, then, we use

$$\zeta_1' y = \zeta_1 + \delta_1 \delta(\zeta_1 - \zeta_0) \quad \text{and} \quad \zeta_0' = \zeta_0 + \delta_2 \delta(\zeta_1 - \zeta_0) \tag{30}$$

as the estimators of the endpoint of the new search domain. Otherwise, if $\delta < -\delta_0$, the following will be used instead:

$$\zeta_1' = \zeta_1 + \delta_2 \delta(\zeta_1 - \zeta_0) \quad \text{and} \quad \zeta_0' = \zeta_0 + \delta_1 \delta(\zeta_1 - \zeta_0). \tag{31}$$

The fact remains that the skew rate is unknown because we would otherwise need to know the global minimizers x^* in advance. Suppose that ξ is a random variable with probability density $p(x) > 0$ on $[a, b]$ and $\xi_1, \ldots, \xi_N$, are samples of ξ. Let $\eta_N = \min_{1 \leq i \leq N} f(\xi_i)$. It is not difficult to see that η_N will tend to $f(x^*) = \min_{a \leq x \leq b} f(x)$ as $N \to \infty$. Moreover, if $f(x)$ has a unique global minimizer x^* on $[a, b]$, then $\xi_N^* \to x^*$ as $N \to \infty$, where ξ_N^* is given by $f(\xi_N^*) = \eta_N$. The above discussion suggests taking

$$\hat{\delta} = \frac{2\xi_N^* - (\zeta_1 + \zeta_0)}{\zeta_1 - \zeta_0} \tag{32}$$

as an estimator for the skew rate δ.

4.2. Multi-Solutions

The Monte Carlo implementation technique in the last section can be extended to the case when the objective function f has multiple global minimizers. The search domain D_k at the k-th iteration can be decomposed into a union of several cuboids of dimension n:

$$D_k = \bigcup_{j=1}^{r_k} D_j^k, \tag{33}$$

so that each smaller cuboid D_j^k can be treated individually as in the above subsection. Usually we assume that for each iteration k, the number r_k is less than an integer m which is given in advance.

5. Constrained and Discreat Minimization

Constrained nonconvex minimization problems arise from broad range of applications. General speaking, solving a constrained minimization problem is much harder than solving an unconstrained problem. Integral global minimization technique using a discontinuous penalty method to convert a constrained minimization problem to an unconstrained one without any constrained qualification requirements. We outline the main ideas of the discontinuous penalty method.

5.1. Discontinuous Penalty Method

We use the discontinuous penalty method to solve a constrained problem:

$$c^* = \min_{x \in S} f(x), \tag{34}$$

where $S \subset X$ is the constrained set.
The discontinuous penalty function associated with S is defined as follows.

Definition. A function $p(x)$ on a metric space (X, d) is a penalty function associated with a constraint set $S \subset X$ if

1. p is lower semicontinuous;
2. $p(x) = 0$ if $x \in S$;
3. $$\inf_{x \notin S_\beta} p(x) > 0,$$

   where $S_\beta = \{ u : d(u, S) \leq \beta \}$, $\beta > 0$, and $d(x, S)$ is the distance from x to the feasible set S defined by

   $$d(x, S) = \inf\{d(x, s) : s \in S\}.$$

Remark. In the above definition we do not require the continuity of p, unlike the traditional definition [20], [7].

Remark. It is expected that the penalty increases when the distance from a point x to the constraint set S increases. We replace the traditional property

$$p(x) > 0, \text{ if } x \notin S$$

by condition 3.

With a penalty function p, we examine a penalized unconstrained minimization problem associated with (34):

$$\min_{x \in X}\{f(x) + \alpha p(x)\}, \tag{35}$$

where α (> 0) is a penalty parameter.

Definition. A penalty function p for the constraint set S is *exact* for (34) if there is a real number $\alpha_0 > 0$ such that for each $\alpha \geq \alpha_0$ we have

$$\min_{x \in X}\{f(x) + \alpha p(x)\} = \min_{x \in S} f(x) = c^* \tag{36}$$

and

$$\{x \in X : f(x) + \alpha p(x) = c^*\} = \{x \in S : f(x) = c^*\} = H^*. \tag{37}$$

We now construct a class of discontinuous penalty functions for the constrained problem (34). Let

$$p_S(x, \delta) = \begin{cases} 0, & x \in S, \\ \delta + d(x), & x \notin S, \end{cases} \tag{38}$$

where δ is a positive number and $d(x)$ is a penalty-like function.

For example, for the inequality-constraint set

$$S = \{x : g_i(x) \leq 0, i = 1, \ldots, r\},$$

we can take

$$d(x) = \sum_{i=1}^{r} \|\max(g_i(x), 0)\|^\rho \text{ or } d(x) = \max_i \|\max(g_i(x), 0)\|^\rho,$$

where $\rho > 0$. If g_i, $i = 1, \ldots, r$, are continuous, then d is continuous.

PROPOSITION 1 *If f is continuous, and d is upper robust on S, or f is upper robust and d is continuous on S, then $f + \alpha p$ is upper robust on S for every $\alpha > 0$.*

THEOREM 3 *[32] The discontinuous penalty function (38) is exact.*

Remark. No constraint qualification is required for the penalty function (38).

Remark. If f is robust piecewise continuous with a robust partition $\{S, S^c\}$, then for each $\alpha > 0$ and $\delta > 0$ the penalized function $f(x) + \alpha\, p_S(x, \delta)$ is a piecewise robust continuous function.

A penalty algorithm is proposed as follows:

Step 1 : Take $c_0 > \min_{u \in S} f(x)$; $\epsilon > 0; n := 0$; $\beta > 1.0$;

$$H_0 = \{\, x \,:\, f(x) + \alpha_0 p(x) \leq c_0 \,\};$$

Step 2 : Calculate the mean value

$$c_{n+1} = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu; \tag{39}$$

Step 3 : Calculate the modified variance

$$v_{n+1} = \frac{1}{\mu(H_n)} \int_{H_n} (f(x) + \alpha_n p(x) - c_n)^2 d\mu.$$

If $v_{n+1} \geq \epsilon$, then $n := n + 1$ and $\alpha_{n+1} = \alpha_n \cdot \beta$, and go to Step 2; otherwise, go to Step 4;

Step 4 : $c^* \Longleftarrow c_{n+1}; H^* \Longleftarrow H_{c_{n+1}}$; Stop.

The algorithm may stop in a finite numbers of iterations, in which case we let $c_{n+k} = c_n$ and $H_{n+k} = H_n, k = 1, 2, \ldots$.

Applying the above algorithm with $\epsilon = 0$, we obtain a decreasing sequence

$$c_1 \geq c_2 \geq \cdots \geq c_n \geq c_{n+1} \geq \cdots \tag{40}$$

and a sequence of sets

$$H_1 \supset H_2 \supset \cdots \supset H_n \supset H_{n+1} \supset \cdots. \tag{41}$$

THEOREM 4 *[32] With this algorithm, we have*

$$\lim_{n \to \infty} c_n = c^* = \min_{u \in S} f(x) \tag{42}$$

and

$$\lim_{n \to \infty} H_n = \bigcap_{k=1}^{\infty} H_n = H^*. \tag{43}$$

5.2. Robustification of Integer and Mixed Programming

A discrete or mixed minimization problem can be robustified to be a problem with a robust piecewise continuous function over a robust set. The following example demonstrates the process.

Example: Consider the following combinatorial optimization problem. Let

$$Z_+^n = \{z = (z^1, \ldots, z^n) : z^i \text{ is a nonnegative integer}, i = 1, \ldots, n\},$$

S be a finite subset of Z_+^n and $f : S \to R^1$ a function defined on S. Let $f(z) = f(z^1, \ldots, z^n)$. The problem is to find the minimum value of f over S:

$$c^* = \min_{z \in S} f(z)$$

and the set of minima

$$H^* = \{z \in S : f(z) = c^*\}.$$

In this case, H^* is nonempty.

We now consider this problem in the space R^n. The set S is not robust in this space. We define

$$D = \{x = (x^1, \ldots, x^n) \in R^n : ([x^1 + 0.5], \ldots, [x^n + 0.5]) \in S\}$$

and

$$F(x) = f([x^1 + 0.5], \ldots, [x^n + 0.5]),$$

where $[a]$ denotes the integer part of the real number a. The set D defined above is a union of n-dimensional cubes, which are robust in R^n. For each real number c, the set $\{x : F(x) < c\}$ is also a union of cubes (or the empty set). Thus, D is a robust set and F is an upper robust function in R^n. Let x^* be a global minimizer of F over D, i.e.,

$$F(x^*) = \min_{x \in D} F(x).$$

Then $x^* \in \operatorname{int} D$ (or one can find a point x_1 in the same cube with x^* such that $x_1 \in \operatorname{int} D$). Therefore, we obtain a robustification of this combinatorial optimization problem. □

6. Numerical Tests

The performance of a global minimization algorithm can only be ascertained by numerical computations on a variety of test problems. There are a lot of test problems for global minimization available in the literature. We select some here and classify them as follows:

(A) Unconstrained or box-constrained minimization.

(C) Constrained minimization.

(D) Discrete minimization, including integer and mixed programming.

The problems selected here represent some well known test problems in global optimization community. The selection range from problems with two variables to problems with a hundred variables, from problems of differentiable objective functions to the problems of a objective function with infinite number of discontinuities; from problems with box constraints to problems with equality and inequality constraints. We also select several discrete or mixed minimization problems. We hope that the selection is general enough to warrant our claim that the integral global optimization technique is powerful, flexible and efficient, and it is competitive with any other existing global optimization algorithms.

All the test problems selected here are solved by packages INTGLOU and INTGLOC, which are the implementations of the algorithms of integral global minimization. The softwares are compiled by MS-FORTRAN 5.1 and are running on MS-DOS environment. These test problems can be solved within a few seconds to a few minutes on an IBM 386/25 personal computer with a math coprocessor.

6.1. Unconstrained or Box Constrained Problems

A set of unconstrained or box constrained test problems are presented in this subsection. We describe each test problem by the following:

1. objective function
2. Search domain (boxed constraints)
3. Solution, including the minimum objective function value computed by the integral global minimization algorithm, the corresponding minimizers.
4. Statistics: we list the number iterations, the number of function evaluations and current value of V_1.

The sources of the problems are also provided. Note that the integral global minimization algorithms do not use any start points.

The stopping criterion employed for all the unconstrained problems selected here is the modified variance $V_1 = 1 \times 10^{-20}$.

Problem A.1. SOURCE: [6].

OBJECTIVE FUNCTION:

$$\begin{aligned} f(x) = \; & [1 + (x_1 + x_2 + 1)^2 \cdot (19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2)] \times \\ & [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]. \end{aligned}$$

SEARCH DOMAIN:

$$D = \{(x_1, x_2) \in R^2 : -2.0 \le x_i \le 2.0, \quad i = 1, 2\}$$

SOLUTION:

$$x^* = (0.0, -1.0) \quad f^* = 3.0.$$

STATISTICS:

1. number of iterations: 19
2. number of function evaluations: 1051
3. current value of modified variance V_1: 9.233×10^{-21}

Problem A.2. SOURCE: [6].
OBJECTIVE FUNCTION:

$$f(x) = 12x_1^2 - 6.3x_1^4 + x_1^6 + 6x_2(x_2 - x_1)$$

SEARCH DOMAIN:

$$D = \{(x_1, x_2) \in R^2 : -10.0 \le x_i \le 10.0, \quad i = 1, 2\}$$

SOLUTION:

$$x^* = (0.0, 0.0) \quad f^* = 2.2497375 \times 10^{-13}$$

STATISTICS:

1. number of iterations: 17
2. number of function evaluations: 951
3. current value of modified variance V_1: $1.1543449 \times 10^{-21}$

Remark. The objective function is so-called *three-hump camel back function.*

Problem A.3. SOURCE: [6].
OBJECTIVE FUNCTION:

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4.$$

SEARCH DOMAIN:

$$D = \{(x_1, x_2) \in R^2 : -2.5 \le x_i \le 2.5, \quad i = 1, 2\}$$

SOLUTION:

$$x^* = (0.08984133, -0.71267531) \text{ and } (-0.08993914, 0.7126753), \quad f^* = -1.031628.$$

STATISTICS:

1. number of iterations: 18
2. number of function evaluations: 931
3. current value of modified variance V_1: 8.216884×10^{-21}

Remark. The objective function is so-called the *six hump camel back function.* It has six minimizers, two maximizers and seven saddle points.

Problem A.4. SOURCE: [8].
Objective Function:

$$f(x) = (1 - 2x_2 + c\sin(4\pi x_2) - x_1)^2 + (x_2 - 0.5\sin(2\pi x_1))^2,$$

where c is a parameter which can be varied to modify the number of extraneous sigularities in the function. Here, we take $c = 0.05$, 0.2, and 0.5.
SEARCH DOMAIN:

$$D = \{(x_1, x_2) \in R^2 : 0.0 < x_1 < 10.0,\ -10.0 < x_2 < 0.0\}$$

SOLUTION: The global minimum value of this problem is 0.0 for each c. The following table presents the numerical approximation of the global minimum value and the minimizers.

	$c = 0.05$	$c = 0.2$	$c = 0.5$
x_1	1.85130447	0.98250584	1.89738692
x_2	−0.40208593	−0.05484892	−0.30049412
f	$1.2122348 \times 10^{-22} \cdot 10^{-32}$	$1.7292806 \cdot 10^{-33}$	$7.7683103 \cdot 10^{-32}$

STATISTICS:

1. number of iterations: 22
2. number of function evaluations: 1660
3. current value of modified variance V_1: $1.0411676 \times 10^{-21}$

Problem A.5. SOURCE: [6].
OBJECTIVE FUNCTION:

$$f(x) = \left(x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos x_1 + 10.$$

SEARCH DOMAIN:

$$D = \{(x_1, x_2) \in R^2 : -5.0 \le x_1 \le 10.0,\ 0.0 \le x_2 \le 15.0\}$$

SOLUTION: The integral global algorithm find three global minimizers in this region:

$$(-3.14159291, 12.275030), \quad (3.141579, 2.274958), \quad (9.424798, 2.474921)$$

with the global minimum value

$$f^* = 0.39788736.$$

STATISTICS:

1. number of iterations: 23
2. number of function evaluations: 1267
3. current value of modified variance V_1: 1.0×10^{-21}

Problem A.6. SOURCE: [6]
OBJECTIVE FUNCTION: Shekel's family (SQRIN)

$$f(x) = -\sum_{i=1}^{m} \frac{1}{(x - a_i)^T (x - a_i) + c_i},$$

where the parameters a_i and c_i are given by the following table:

i	a_i				c_i
1	4.0	4.0	4.0	4.0	0.1
2	1.0	1.0	1.0	1.0	0.2
3	8.0	8.0	8.0	8.0	0.2
4	6.0	6.0	6.0	6.0	0.4
5	3.0	7.0	3.0	7.0	0.4
6	2.0	9.0	2.0	9.0	0.6
7	5.0	5.0	3.0	3.0	0.3
8	8.0	1.0	8.0	1.0	0.7
9	6.0	2.0	6.0	2.0	0.5
10	7.0	3.6	7.0	3.6	0.5

SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_4) \in R^4 : 0.0 \le x_i \le 10.0,\ i = 1, \ldots, 4\}.$$

SOLUTIONS:
SHEKEL 5:

$$x^* = (4.00003727, 4.00013375, 4.00003730, 4.00013346), \quad f^* = -10.153200.$$

SHEKEL 7:

$$x^* = (4.00057280, 4.00069020, 3.99948997, 3.99960620), \quad f^* = -10.402941.$$

SHEKEL 10:

$$x^* = (4.00074671, 4.00059326, 3.99966290, 3.99950981), \quad f^* = -10.536410.$$

STATISTICS: SHEKEL 5

1. number of iterations: 41
2. number of function evaluations: 2453
3. current value of modified variance V_1: $1.7979744 \times 10^{-21}$

SHEKEL 7

1. number of iterations: 42
2. number of function evaluations: 3028
3. current value of modified variance V_1: 1.0×10^{-21}

SHEKEL 10

1. number of iterations: 41
2. number of function evaluations: 2735
3. current value of modified variance V_1: 1.0×10^{-21}

Problem A7 SOURCE: [6]
Objective Function:

$$f(x) = -\sum_{i=1}^{m} c_i \exp\left(-\sum_{j=1}^{n} a_{ij}(x_j - p_{ij})^2\right).$$

where $x = (x_1, \ldots, x_n)$, and the parameters are given in the following tables:

HARTM 3 : $m = 4,\ n = 3$

i	a_{ij}			c_i	p_{ij}		
1	3.0	10.0	30.0	1.0	0.3689	0.1170	0.2673
2	0.1	10.0	35.0	1.2	0.4699	0.4387	0.7470
3	3.0	10.0	30.0	3.0	0.1091	0.8732	0.5547
4	0.1	10.0	35.0	3.2	0.03815	0.5743	0.8828

HARTM 6 : $m = 4,\ n = 6$

i	a_{ij}						c_i
1	10.0	3.0	17.0	3.5	1.7	8.0	1.0
2	0.05	10.0	17.0	0.1	8.0	14.0	1.2
3	3.0	3.5	1.7	10.0	17.0	8.0	3.0
4	17.0	8.0	0.05	10.0	0.1	14.0	3.2

i	p_{ij}					
1	0.1312	0.1696	0.5569	0.0124	0.8283	0.5886
2	0.2329	0.4135	0.8307	0.3736	0.1004	0.9991
3	0.2348	0.1451	0.3522	0.2883	0.3047	0.6650
4	0.4047	0.8828	0.8732	0.5743	0.1091	0.0381

SOLUTIONS:

HARTM 3

$$x^* = (0.11461478, 0.55564892, 0.85254688),\quad f^* = -3.8627821.$$

HARTM 6

$$x^* = (0.20169, 0.15001, 0.47687, 0.27533, 0.31165, 0.65730),\quad f^* = -3.322368.$$

STATISTICS:

HARTM 3

1. number of iterations: 23
2. number of function evaluations: 1150
3. current value of modified variance V_1: 1.0×10^{-21}

HARTM 6

1. number of iterations: 49
2. number of function evaluations: 3345
3. current value of modified variance V_1: 1.0×10^{-21}

Problem A.8. SOURCE: [9] with an enlarged search domain.
OBJECTIVE FUNCTION:

$$f(x) = \sum_{i=1}^{11} \left(a_i - x_1 \frac{b_i^2 + b_i x_2}{b_i^2 + b_i x_3 + x_4} \right)^2,$$

where a_i and b_i, $i = 1, \ldots, 11$ are given as follows:

i	a_i	$1/b_i$
1	0.1957	0.25
2	0.1947	0.5
3	0.1735	1
4	0.1600	2
5	0.0844	4
6	0.0627	6
7	0.0456	8
8	0.0342	10
9	0.0323	12
10	0.0235	14
11	0.0246	16

SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_4) \in R^4 : -0.3 \leq x_i \leq 0.3, \;\; i = 1, \ldots, 4\}$$

SOLUTION:

$$x^* = (0.19282941, 0.19095407, 0.12315108, 0.13581648), \;\; f^* = 3.0748802 \times 10^{-4}.$$

STATISTICS:

1. number of iterations: 54
2. number of function evaluations: 7592
3. current value of modified variance V_1: 1.0×10^{-21}

Problem A.9. SOURCE: [17] with an enlarged search domain.
OBJECTIVE FUNCTION:

$$f(x) = \sum_{i=1}^{81} R_i^2$$

where

$$R_i = x_1 \exp\left\{ - \left[\frac{z_i - x_3}{x_5} \right]^2 \right\} + x_2 \exp\left\{ - \left[\frac{z_i - x_4}{x_6} \right]^2 \right\} - y_i$$

and

$$z_i = 4.0 + 0.1(i+1), \quad i = 1, 2, \ldots, 81,$$

$$y_i = 130.89 \exp\left\{-\left[\frac{z_i - 6.73}{1.2}\right]^2\right\} + 52.6 \exp\left\{-\left[\frac{z_i - 9.342}{0.97}\right]^2\right\}, \quad i = 1, 2, \ldots, 81.$$

SEARCH DOMAIN:

$$D = \left\{ \begin{array}{l} 120 \le x_1 \le 150, \ 30 \le x_2 \le 70, \ 4 \le x_3 \le 10, \\ 5 \le x_4 \le 15, \ 0.5 \le x_5 \le 4, \ 0.2 \le x_6 \le 2 \end{array} \right\}.$$

SOLUTION:

$$x_1 = 130.89, \ x_2 = 52.59, \ x_3 = 6.73, \ x_4 = 9.342, \ x_5 = 1.2, \ x_6 = 0.97,$$

$$f^* = 1.6383836 \times 10^{-10}$$

STATISTICS:

1. number of iterations: 77
2. number of function evaluations: 5187
3. current value of modified variance V_1: 1.0×10^{-21}

We can consider a minimization of a function

$$f(x) = \sum_{i=1}^{81} |R_i|$$

or

$$f(x) = \max_{i=1,\ldots,81} |R_i|$$

with the same search domain.

Problem A.10. SOURCE: [14]. OBJECTIVE FUNCTION:

$$f(x) = -\left(\sum_{i=1}^{8} x_i^2\right) \times \left(\sum_{i=1}^{8} x_i^4\right) + \left(\sum_{i=1}^{8} x_i^3\right)^2$$

SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_8) \in R^8 : 0.0 \le x_i \le 1.0, \ i = 1, \ldots, 8\}$$

Problem A.11. SOURCE: [14].
OBJECTIVE FUNCTION:

$$f(x) = \frac{\pi}{n}\{\sin^2(\pi x_1) + \sum_{i=1}^{n-1}(x_i - 1.0)^2[1 + 10.0\sin^2(\pi x_{i+1})] + (x_n - 1.0)^2\}$$

SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_n) \in R^n : -10.0 \leq x_i \leq 10.0, \quad i = 1, \ldots, n\}$$

SOLUTION:

$$x^* = (1, \ldots 1) \quad f^* = 0.$$

The following tableau gives the number of iterations N_i, the amount of function evaluation N_f, the function value f^* and the current value of modified variance V_1 corresponding the cases of number of variables $n = 5$, 10, 20, 50, respectively.
The stopping criterion for this problem is $V_1 < 10^{-25}$.

n	5	10	20	50	100
N_i	52	93	172	380	863
N_f	2765	5276	12376	49359	128483
f^*	$1.076 \cdot 10^{-13}$	$6.43 \cdot 10^{-13}$	$1.65 \cdot 10^{-12}$	$3.41 \cdot 10^{-12}$	$2.90 \cdot 10^{-12}$
V_1	$4.12 \cdot 10^{-26}$	$8.77 \cdot 10^{-26}$	$7.07 \cdot 10^{-26}$	$8.18 \cdot 10^{-26}$	$9.71 \cdot 10^{-26}$

Problem A.12. SOURCE: [14] with modification.
OBJECTIVE FUNCTION:

$$g(x) = \sin^2(3\pi x_1) + \textstyle\sum_{i=1}^{n-1}(x_i - 1.0)^2[1.0 + \sin^2(3\pi x_{i+1})] + \\ (x_n - 1.0)^2[1.0 + \sin^2(2\pi x_n)], \quad f(x) = g(x) + \frac{[g(x)]}{n},$$

where $[y]$ denote the integer part of y. Thus, the objective function f is discontinuous. SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_n) \in R^n : -10.0 \leq x_i \leq 10.0, \quad i = 1, \ldots, n\}$$

SOLUTION:

$$x^* = (1.0, \ldots 1.0) \quad f^* = 0.$$

The following tableau gives the number of iterations N_i, the amount of function evaluation N_f, the minimum function value f^* and the current values of the modified variance V_1 corresponding cases of number of variables $n = 5$, 10, 20, 50, respectively. The stopping criterion for this problem is $V_1 < 10^{-25}$.

n	5	10	20	50
N_i	56	101	186	412
N_f	3208	5996	12549	54734
f^*	$5.838578 \cdot 10^{-14}$	$6.414436 \cdot 10^{-13}$	$1.180750 \cdot 10^{-12}$	$2.285634 \cdot 10^{-12}$
V_1	$4.986358 \cdot 10^{-26}$	$5.835348 \cdot 10^{-26}$	$5.241681 \cdot 10^{-26}$	$8.942764 \cdot 10^{-26}$

Problem A.13. SOURCE: [4]
OBJECTIVE FUNCTION:

$$f(x) = \begin{cases} 1.0 + \frac{\sum_{i=1}^{n} |x_i|}{n} + \operatorname{sgn}\left(\sin(\frac{n}{\sum_{i=1}^{n} |x_i|}) - 0.5\right), & x \neq 0, \\ 0, & x = 0 \end{cases} \tag{44}$$

SEARCH DOMAIN:

$$D = \{(x_1, \ldots, x_n) : -1.0 \leq x_i \leq 1.0,\ i = 1, \ldots, n\}$$

SOLUTION:

$$x^* = (0, \ldots 0), \quad f^* = 0.$$

Remark. The function has an infinite number of discontinuous hypersurfaces. Its unique global minimizer is at the origin where the objective function has a discontinuity of "the second kind." Since the restriction of the variable value that sine function can take, the function f takes the value zero when $\sum_{i=1}^{n} |x_i|/n < 10^{-9}$. The following tableau gives the data of this text problem.

n	5	10	20	50
N_i	77	128	226	711
N_f	5203	10223	25527	105747

6.2. Constrained Minimization Problems

We present a set constrained problems in this subsection. We describe each test problem by the following format:

1. Objective function.

2. Constraints, including constrain functions and boxed constraints.

3. Solution, the minimum objective function value computed by the integral global minimization algorithm, the corresponding minimizers.

4. Statistics, including the number of iterations, the number of function evaluations and the current value of modified variance V_1.

The discontinuous penalty method presented in Section 5 is used to solve all the constrained problems in this subsection.

Unless otherwise stated explicitly, the stopping criterion used in the programs for solving all numerical tests in this subsection is 1.0×10^{-15}.

Problem C.1. SOURCE: [4].

OBJECTIVE FUNCTION:

$$f(x) = 100(x_2 - x_1)^2 + (1 - x_1)^2$$

CONSTRAINTS:

$$h(x) = x_1^2 - x_1 + x_2 - 0.9 = 0, \ \ -1.0 \le x_1, \ \ x_2 \le 1.0.$$

SOLUTION:

$$x^* = (0.965932, 0.932907) \quad \text{and} \quad f^* = 0.001162$$

with

$$h(x^*) = 2.109617 \cdot 10^{-13}.$$

The penalty function

$$p(x) = \alpha |h(x)|^{1.8}, \ \ \alpha = 1000$$

is used to solve this minimization problem.

STATISTICS:

1. number of iterations: 31;

2. number of function evaluations: 2829;

3. current value of modified variance V_1: 4.05785×10^{-16}.

Problem C.2. SOURCE: [8].

OBJECTIVE FUNCTION:

$$f(x) = -x_1 - x_2 + x_3$$

CONSTRAINTS:

$$\sin(4\pi x_1) - 2\sin^2(2\pi x_2) - 2\sin^2(2\pi x_3) \ge 0, \ \ -5 \le x_1, \ \ x_2 \le 5.$$

SOLUTION:

$$x^* = (4.75, 5.0, -5.0), \quad \text{and} \quad f^* = -14.75.$$

STATISTICS:

1. number of iterations: 49;
2. number of function evaluations: 4440;
3. current value of modified variance V_1: 0.

Problem C.3. SOURCE: [38].
OBJECTIVE FUNCTION:

$$f(x) = -2x_1^2 - x_1x_2 - 2x_2.$$

CONSTRAINTS:

$$x_1 + x_2 \leq 1, \ \ 1.5x_1 + x_2 \leq 1.4,$$
$$0.0 \leq x_1 \leq 10.0, \ \ -10.0 \leq x_2 \leq 0.0.$$

SOLUTION:

$$x^* = (7.6, -10), \qquad f^* = -19.52.$$

STATISTICS:

1. number of iterations: 43;
2. number of function evaluations: 3914;
3. current value of modified variance V_1: 4.94434×10^{-16}.

Remark. This is a counterexample to Ritter's method [22]. The global minimizer will not be found by Ritter's method unless one happens to begin with $(7.6, -10)$ as the first local optimum.

Problem C.4. SOURCE: [38].
OBJECTIVE FUNCTION:

$$f(x) = -x_1^2 - x_2^2 - (x_3 - 1)^2.$$

CONSTRAINTS:

$$x_1 + x_2 - x_3 \leq 0, \ \ -x_1 + x_2 - x_3 \leq 0, \ \ 12x_1 + 5x_2 + 12x_3 \leq 22.8,$$
$$12x_1 + 12x_2 + 7x_3 \leq 17.1, \ \ -6x_1 + x_2 + x_3 \leq 1.9,$$
$$-10.0 \leq x_1 \leq 10.0, \ \ 0.0 \leq x_2 \leq 10.0, \ \ 10.0 \leq x_3 \leq 10.0.$$

SOLUTION:

$$x^* = (3.42, 0, -3.42), \quad f^* = -31.2328.$$

STATISTICS:

1. number of iterations: 74;

2. number of function evaluations: 8876;

3. current value of modified variance V_1: 4.48476×10^{-16}.

Remark. This is a counterexample to Tuy's method [26]. A local optimum occurs at the vertex $x^0 = (0,0,0)$ with $f(x^0) = -1$; Tuy's method will produces an infinite cycling and the process does not terminate.

Problem C.5. SOURCE: [38].
OBJECTIVE FUNCTION:

$$f(x) = -(x_1 - 1)^2 - x_2^2 - (x_3 - 1)^2.$$

CONSTRAINTS:

$$\begin{aligned}
&x_1 + x_2 - x_3 \le 1, \quad -x_1 + x_2 - x_3 \le -1, \\
&12x_1 + 5x_2 + 12x_3 \le 34.8, \quad 12x_1 + 12x_2 + 7x_3 \le 17.1, \\
&-6x_1 + x_2 + x_3 \le -4.1, \quad 0.0 \le x_1,\ x_2,\ x_3, \le 5.0.
\end{aligned}$$

SOLUTION:

$$x^* = (1,0,0), \quad f^* = -1.$$

STATISTICS:

1. number of iterations: 37;

2. number of function evaluations: 2043;

3. current value of modified variance V_1: 7.66012×10^{-16}.

Problem C.6. SOURCE: [12].
OBJECTIVE FUNCTION:

$$f(x) = (x_1^4 + x_2 + x_3) - (x_1 + x_2^2 - x_3)^2.$$

CONSTRAINTS:

$$\begin{aligned}
&(x_1 - x_2 - 1.2)^2 + x_2 \le 4.4, \quad x_1 + x_2 + x_3 \le 6.5, \\
&1.4 \le x_1 \le 5.0, \quad 1.6 \le x_2 \le 5.0, \quad 1.8 \le x_3 \le 5.0.
\end{aligned}$$

SOLUTION:

$$x^* = (1.4, 1.809502, 1.8), \quad f^* = 4.576804.$$

STATISTICS:

1. number of iterations: 39;

2. number of function evaluations: 2111;

3. current value of modified variance V_1: 8.17440×10^{-16}.

Problem C.7. SOURCE: [10].

OBJECTIVE FUNCTION:

$$f(x) = f_1(x_1) + f_2(x_2),$$

where

$$f_1(x_1) = \begin{cases} 30x_1, & 0 \leq x_1 < 300, \\ 31x_1, & 300 \leq x_1 < 400, \end{cases} \qquad f_2(x_2) = \begin{cases} 28x_2, & 0 \leq x_2 < 100, \\ 29x_2, & 100 \leq x_2 < 200, \\ 30x_2, & 200 \leq x_2 < 1000. \end{cases}$$

CONSTRAINTS:

$$\begin{gathered} x_1 = 300 - \frac{x_3x_4}{131.078}\cos(1.48577 - x_6) + \frac{0.90798x_3^2}{131.078}\cos(1.47588), \\ x_2 = -\frac{x_3x_4}{131.078}\cos(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078}\cos(1.47588), \\ x_5 = -\frac{x_3x_4}{131.078}\sin(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078}\sin(1.47588), \\ 200 - \frac{x_3x_4}{131.078}\sin(1.48477 - x_6) + \frac{0.90798}{131.078}x_3^2\sin(1.47588) = 0, \\ 0 \leq x_1 \leq 400, \ 0 \leq x_2 \leq 1000, \ 340 \leq x_3 \leq 420, \\ 340 \leq x_4 \leq 420, \ -1000 \leq x_5 \leq 1000, \ 0 \leq x_6 \leq 0.5236. \end{gathered}$$

SOLUTION:

$$x^* = (202.99666, 100.0, 383.07092, 419.99999, -10.90767, 0.073148)$$

$$f^* = 8889.8999$$

STATISTICS:

1. number of iterations: 56;
2. number of function evaluations: 5893;
3. current value of modified variance V_1: 6.18995×10^{-16}.

Remark. The objective of this test problem is a discontinuous robust function with four nonlinear equality constraints. We take x_3 and x_6 as independent variables. Then x_1, x_2, x_4 and x_5 are functions of x_3 and x_6. Thus, in addition to the box constraints on these independent variables, there are 8 more nonlinear inequality constraints. The discontinuous penalty function is applied to these inequality constraints.

Problem C.8. SOURCE: [21].
OBJECTIVE FUNCTION:

$$\begin{aligned} f(x) &= 0.0204x_1x_4(x_1+x_2+x_3)+0.0187x_2x_3(x_1+1.57x_2+x_4)+ \\ &\quad 0.0607x_1x_4x_5^2(x_1+x_2+x_3)+0.0437x_2x_3x_6^2(x_1+1.57x_2+x_4), \end{aligned}$$

subject to the inequality constraints:

$$\begin{aligned} x_i \geq 0, \quad i=1,\ldots,6, \\ g_1(x) = x_1x_2x_3x_4x_5x_6 - 2070 \geq 0, \\ g_2(x) = 1 - 0.00062x_1x_4x_5^2(x_1+x_2+x_3) \\ -0.0058x_2x_3x_6^2(x_1+1.57x_2+x_4) \geq 0 \end{aligned}$$

The problem was solved by Ballard, Jelink and Schinzinger [3]. The minimization process starts with a feasible point:

$$\begin{aligned} x_1 = 5.54,\ x_2 = 4.4,\ x_3 = 12.02, \\ x_4 = 11.82,\ x_5 = 0.702,\ x_6 = 0.852 \end{aligned}$$

and leads to a solution

$$\begin{aligned} x_1 = 5.3336,\ x_2 = 4.6585,\ x_3 = 10.4365, \\ x_4 = 12.0840,\ x_5 = 0.7525,\ x_6 = 0.8781. \end{aligned}$$

The objective function value at the solution is $f^* = 135.1155$. Price [21] resolved the problem with the controlled random search method and suggested that it be used as a test problem of constrained global minimization.

The following solution is obtained by the integral global minimization with the discontinuous penalty technique in a large search region D:

$$D = \{x \in R^6 : 0.0 \leq x_i \leq 20.0,\ \ i=1,\ldots,6\}.$$

$$\begin{aligned} x_1 = 5.41411876,\ x_2 = 4.71604587,\ x_3 = 10.34384982, \\ x_4 = 11.88555219,\ x_5 = 0.74910661,\ x_6 = 0.88027699, \end{aligned}$$

and

$$f^* = 135.09767268.$$

STATISTICS:

1. number of iterations: 599;
2. number of function evaluations: 87475;
3. current value of modified variance V_1: 3.03333×10^{-16}.

Remark. The solution x^* is very closed to the boundary of constraints:

$$g_1(x^*) = 9.9685 \cdot 10^{-8}, \quad \text{and} \quad g_2(x^*) = 1.5982 \cdot 10^{-10}.$$

Problem C.9. *Source*: [13].
OBJECTIVE FUNCTION:

$$\begin{aligned} f(x) &= 0.7854x_1x_2^2(3.3333x_3^2 + 14.9334x_3 - 43.0934) - 1.5080x_1(x_6^2 + x_7^2) \\ &\quad +7.4770(x_6^3 + x_7^3) + 0.7854(x_4x_6^2 + x_5x_7^2). \end{aligned}$$

CONSTRAINTS:

$$\begin{gathered} x_1x_2^2x_3 \geq 27, \quad x_1x_2^2x_3^2 \geq 397.5, \\ x_2x_3x_6^4/x_4^3 \geq 1.93, \quad x_2x_3x_7^4/x_5^3 \geq 1.93, \\ \frac{10\sqrt{\left[\frac{745x_4}{x_2x_3}\right]^2 + 16.91 \cdot 10^6}}{x_6^3} \leq 1100, \quad \frac{10\sqrt{\left[\frac{745x_5}{x_2x_3}\right]^2 + 157.5 \cdot 10^6}}{x_7^3} \leq 850, \\ x_2x_3 \leq 40, \quad 5 < x_1/x_2 \leq 12, \quad 1.5x_6 + 1.9 \leq x_4, \\ 1.1x_7 + 1.9 \leq x_5, \quad 2.6 \leq x_1 \leq 3.6, \quad 0.7 \leq x_2 \leq 0.7, \\ 17 \leq x_3 \leq 28, \quad 7.3 \leq x_4 \leq 8.3, \quad 7.3 \leq x_5 \leq 8.3, \\ 2.9 \leq x_6 \leq 3.9, \quad 5.0 \leq x_7 \leq 5.5. \end{gathered}$$

SOLUTION:

$$x^* = (3.5, 0.7, 17.0, 7.30, 7.72, 3.35, 5.29), \quad f^* = 2994.42.$$

STATISTICS

1. number of iterations: 128;
2. number of function evaluations: 8839;
3. current value of modified variance V_1: 2.22273×10^{-16}.

Problem C.10. SOURCE: [23].
OBJECTIVE FUNCTION:

$$f(x) = 1.10471x_1^2x_2 + 0.04811x_3x_4(14 + x_2)$$

CONSTRAINTS:

$$\begin{gathered} g_1(x) = x_4 - x_1 \geq 0, \\ g_2(x) = \tfrac{13600}{10^6}\sqrt{t_1^2 + \tfrac{2t_1t_2x_2}{\sqrt{x_2^2 + (x_1 + x_3)^2}} + t_2^2}/10^6 \geq 0, \\ g_3(x) = 3 - \tfrac{5.04}{x_4x_3^2} \geq 0, \end{gathered}$$

$$g_4(x) = \frac{4.013}{1.96\times 10^8}\sqrt{EG}\left(1 - \frac{x_3}{28}\sqrt{\frac{E}{G}}\right) \geq 0.006,$$
$$g_5 = 0.25 - \frac{2.1952}{x_4 x_3^3} \geq 0,$$
$$t_1 = 6000/(1.414x_1x_2),\ \ E = x_3x_4^3 10^7/4,\ \ G = 4x_3x_4^3 10^6,$$
$$t_2 = 3000(14 + x_2/2)\sqrt{x_2^2 + (x_1 + x_3)^2}/J,$$
$$J = 0.707x_1x_2\left(\frac{x_2^2}{6} + \frac{(x_1+x_3)^2}{2}\right),$$
$$0.125 \leq x_1 \leq 20.0,\quad 0.0 \leq x_2 \leq 20.0,\quad 0.0 \leq x_3 \leq 20.0,\quad 0.0 \leq x_4 \leq 20.0.$$

SOLUTION:

$$x^* = (0.15321, 16.93611, 3.00768, 0.32293),\ \text{ and }\ f^* = 1.88446227.$$

STATISTICS:

1. number of iterations: 159;
2. number of function evaluations: 23202;
3. current value of modified variance V_1: 5.81420×10^{-11}.

Remark. A solution was reported in [23] with $f^* = 2.38116$. Here, we find a different feasible solution with significantly better objective function value.

6.3. Discrete and Mixed Minimization Problems

Robustification technique enables us to treat discrete and mixed programming problems as continuous ones. In this subsection, we present several discrete or mixed test problems. The integral global approach with discontinuous penalty method is applied to solve these problems. The format of the descriptions of the problems is the same as the previous subsection.

Problem D.1. SOURCE: [4].

OBJECTIVE FUNCTION: SOURCE: [6] with discrete constraints.

$$\begin{aligned} f(x) = \ & [1 + (x_1 + x_2 + 1)^2 \cdot (19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2)]\times \\ & [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]. \end{aligned}$$

CONSTRAINTS:

$$D = \{(x_1, x_2) : x_1, x_2 = 0.001i,\ \ i = -2000, -1999, \ldots, 1999, 2000\}.$$

SOLUTION:

$$x^* = (0.000, -1.000) \quad f^* = 3.0.$$

STATISTICS:

1. number of iterations: 9;
2. number of function evaluations: 291;
3. current value of modified variance V_1: 0.

Problem D.2. SOURCE: [1], [27].
OBJECTIVE FUNCTION:

$$\sum_{i=1}^{n} \frac{a_i}{x_i},$$

where $n = 3$, $a_1 = 33.7539$, $a_2 = 1.4430$ and $a_3 = 1.3885$.
CONSTRAINT:

$$\sum_{i=1}^{n} x_i = M, \quad 1 \leq x_i \leq N_i, \quad x_i \text{ is integer}, \ i = 1, \ldots, n,$$

where $N_1 = 16$, $N_2 = 20$, $N_3 = 28$, and $M = 24$.
SOLUTION:

$$x^* = (16, 4, 4) \quad \text{and} \quad f^* = 2.8150.$$

STATISTICS:

1. number of iterations: 5;
2. number of function evaluations: 171;
3. current value of modified variance V_1: 0.

Problem D.3. SOURCE [16]
OBJECTIVE FUNCTION:

$$f(x) = (x_1 - 3)^2 + (x_2 - 2)^2 + (x_3 + 4)^2.$$

CONSTRAINTS:

$$g_1 = x_1 + x_2^2 + x_3^{0.5} - 10 \geq 0.0, \quad g_2 = \frac{x_1^2}{4.166} - x_2 + \frac{x_3}{3.921} + 3 \geq 0.0,$$
$$g_3 = -4x_1 + x_2^2 + x_3^{-3.5} + 12 \geq 0.0, \quad x_3 \geq 0, \quad x_1 \text{ and } x_2 \text{ are integers.}$$

SOLUTION:

$$x^* = (3, 3, 0.0) \quad \text{and} \quad f^* = 17.0.$$

STATISTICS:

1. number of iterations: 23;
2. number of function evaluations: 1228;

3. current value of modified variance V_1: 2.48615×10^{-16}.

Remark. It was reported in [16] that the problem has minimizer $x^* = (4, 3, 0.598)$ with the function value $f^* = 23.141604$. In Loh's dissertation [15], the constraints have been changed to: $\bar{g}_i \geq 0.1$, $i = 1, 2, 3$, where $\bar{g}_1 = x_1 + 2x_2 + x_3^{0.5} - 1.0$ and $\bar{g}_i = g_i$, $i = 2, 3$. Let us name this new problem as D3A. A solution of D3A given in [15] was $x^* = (4, 3, 0.631)$ with the function value $f^* = 23.45$.

The solution of D3A obtained by the integral global minimization algorithm is $x^* = (4, 3, 0.1)$ with the function value $f^* = 18.81$. The following is the related statistics:

STATISTICS OF D3A:

1. number of iterations: 29;
2. number of function evaluations: 1891;
3. current value of modified variance V_1: 5.95563×10^{-16}.

Problem D.4. SOURCE: [5].

OBJECTIVE FUNCTION:

$$f(x) = -x_3 - x_4 - x_5.$$

CONSTRAINTS:

$$\begin{gathered} 20x_1 + 30x_2 + x_3 + 2x_4 + 2x_5 \leq 180, \quad 30x_1 + 20x_2 + 2x_3 + x_4 + 2x_5 \leq 150, \\ -60x_1 + x_3 \leq 0, \quad -75x_2 + x_4 \leq 0, \quad 0 \leq x_i \leq 1, \qquad i = 1, 2, \\ 0 \leq x_i \leq 75, \qquad i = 3, 4, 5, \quad x_i \text{ integer} \qquad i = 1, \ldots, 5. \end{gathered}$$

SOLUTION:

$$x^* = (1, 1, 24, 52, 0) \qquad f^* = -76.$$

STATISTICS:

1. number of iterations: 14;
2. number of function evaluations: 1486;
3. current value of modified variance V_1: 0.

Remark. There are at least six alternative global minimizers. After 1131 function evaluations, the global minimizer is found. The variance does not equal zero until 1486 function evaluations.

Problem D.5. SOURCE: [5].

OBJECTIVE FUNCTION:

$$f(x) = x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_6x_7x_8 + x_2x_5x_7.$$

CONSTRAINTS:

$$2x_1 + 2x_4 + 8x_8 \geq 12,\ 11x_1 + 7x_4 + 13x_6 \geq 41,\ 6x_2 + 9x_4x_6 + 5x_7 \geq 60,$$
$$3x_2 + 5x_5 + 7x_8 \geq 42,\ 6x_2x_7 + 9x_3 + 5x_5 \geq 53,$$
$$4x_3x_7 + x_5 \geq 13,\ 2x_1 + 4x_2 + 7x_4 + 3x_5 + x_7 \leq 69,$$
$$9x_1x_8 + 6x_3x_5 + 4x_3x_7 \leq 47,\ 12x_2 + 8x_2x_8 + 2x_3x_6 \leq 73;$$
$$x_3 + 4x_5 + 2x_6 + 9x_8 \leq 31,\ x_i \leq 7,\ i = 1,3,4,6,8,$$
$$x_i \leq 15,\ i = 2,5,7,\ x_i \text{ integer } i = 1,\ldots,8.$$

SOLUTION:

$$x^* = (5,4,1,1,6,3,2,0) \quad f^* = 110.$$

Remark. This is the most difficult one among the five test problems presented in [5]. After 919 function evaluations, the global minimizer is found. The variance does not equal to zero until 1370 function evaluations.

STATISTICS:

1. number of iterations: 15;
2. number of function evaluations: 1370;
3. current value of modified variance V_1: 0.

Problem D.6. SOURCE: [10].
OBJECTIVE FUNCTION:

$$f(x) = 5.3578547x_3^2 + 0.835689x_1x_5 + 37.293239x_1 - 40792.141.$$

CONSTRAINTS:

$$0 \leq 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 \leq 92,$$
$$90 \leq 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 \leq 110,$$
$$20 \leq 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 \leq 25,$$
$$78 \leq x_1 \leq 102,\ 23 \leq x_2 \leq 45,\ x_1, x_2 \text{ are integers},\ 27 \leq x_i \leq 45,\ i = 3,4,5.$$

SOLUTION:

$$x^* = (78, 33, 29.99525603, 45.0, 36.77581291) \quad f^* = -30665.53867176.$$

STATISTICS:

1. number of iterations: 98;
2. number of function evaluations: 11849;
3. current value of modified variance V_1: 5.55430×10^{-16}.

Remark. In [5], the problem was restated as a mixed programming problem.

7. Conclusions

The fundamental theory of integral global optimization is based on robust analysis and Q-measure theory. The theory provides a set of necessary and sufficient conditions to characterize global minimizers and suggests an intuitive approach to locate the global minimizers. The theory is mathematically sound and is well received in mathematics community.

The detailed accounts of the implementation of integral global approach for solving unconstrained minimization problems is presented. The discontinuous penalty method and robustification technique provide an unified approach to solve unconstrained problems, constrained problems, continuous, discrete or mixed problems. Most remarkably, the discontinuous penalty method is exact, and there is no constrained qualification requirements for the method. The collection of numerical tests presented here illustrate the effectiveness of this unified approach.

There are many different algorithms available to solve unconstrained, constrained or discrete, mixed optimization problems. Some of them, based on gradient methods or others, may have better performance than the integral approach for some problems with special structures. However, to the best of our knowledge, there is no method which is both flexible enough to handle discontinuous problems or discrete problems in a unified fashion, and very efficient. We are confident that the integral global optimization is a valuable addition to ever growing global optimization techniques.

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